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A MINIMISATION METHOD OF BOOLEAN FUNCTIONS

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ABSTRACT. Algebraic, graphical, chart and numerical methods of minimisation of Boolean Functions have been proposed by several authors (Karnaugh, 1953; McClusky 1956; Shannon, 1938; staff of Harvard computation laboratory, 1951 and Troye, 1959). These methods aim at improving the design of switching circuits which are extensively employed now a days in digital equipments. In the present paper we shall put forward a method of minimisation which has been developed by a combination of the principles underlying the methods of Svoboda and McClusky. In particular, a procedure has been suggested which reduces to a great extent the number of trial-repetitions required to minimise functions which form into a cyclic basic cell chart.

INTRODUCTION

(a) *The minimal form*

Functions of Boolean variables, like the variables themselves, express the states of binary quantities. The function can have only two values 1 or 0, corresponding to the presence or absence of a particular state. A convenient way of describing the function is to specify in a table the value of the function for each combination of input conditions, such as shown in Table 1, for a three variable function $f(x_1, x_2, x_3)$.

TABLE I

x_1	x_2	x_3	f	Minterms	Decimal equivalent values of minterms
0	0	0	1	$x'_1x'_2x'_3$	0
0	0	1	0	$x'_1x'_2x_3$	1
0	1	0	0	$x'_1x_2x'_3$	2
0	1	1	1	$x'_1x_2x_3$	3
1	0	0	1	$x_1x'_2x'_3$	4
1	0	1	0	$x_1x'_2x_3$	5
1	1	0	0	$x_1x_2x'_3$	6
1	1	1	1	$x_1x_2x_3$	7

In canonical form, f can be expressed as

$$f = x'_1x'_2x'_3+x'_1x_2x_3+x_1x'_2x'_3+x_1x_2x_3$$
$$= \Sigma\,(0, 3, 4, 7)[\text{decimal mode of writing } f]$$

The expression for f derived from the truth table, called the canonical expansion of f, can be written in several reduced forms by applying few theorems of Boolean Algebra. But when the function becomes complex, algebraic manipulation is not very helpful and we resort to special methods.

Of all the alternative expressions for f, we call one (and sometimes more than one) the "minimal form" which involves a minimum number of total operations. The number of operations equals the sum total of "Boolean product" operations to realise the reduced minterms and the "Boolean sum" operations to realise the expansion. It follows from this definition that a minimum of diode circuit elements will be required for physical realisation of the function expressed in the minimal form.

(b) *Neighbours, cells and weight*

The terms of the canonical form of a given n-variable Boolean function can be depicted by nodes of an n-dimensional cube. A single variable is depicted by two modes connected by a line. With two variables (x_1, x_2) we require four nodes connected by four lines. When there are n variables, we take $(n+1)$ vertical dotted line, and number then 0, 1, 2, ... n from left. We construct n_{c_0}, n_{c_1}, ... n_{cn} number of nodes over each of these lines respectively and name the nodes with binary numbers having no '1', one '1', two '1's, and so on up to n '1's, i.e. having index values $r = 0, 1, 2 \ldots, n$, over the $(n+1)$ vertical lines. These are then arranged from top to bottom over any line having increasing decimal equivalent value. Lines are then drawn between nodes which differ in exactly one variable and no line is drawn further. The vertices of the cube represent all possible minterms of the canonical expansion and the cube can be considered to be made up of cells (Urbano and Mueller, 1956).

0-cell or vertex	-a point	$k = 0$
1-cell	-a line	$k = 1$
2-cell	-quadrilateral	$k = 2$
3-cell	-hexahedron	$k = 3$
k-cell		$k = k$

where k denotes the order of the cells.

The nodes or vertices which are joined to a particular node or vertex, are called the neighbours or adjacent states of the vertex. The total number of neighbours belonging to the body of the specified Boolean function with reference to a particular vertex is called the weight of the vertex, and this equals to the number of 1-cells incident with the vertex.

To illustrate the above terms let us take the function expressed in canonical form as

$$f(x_5, x_4, x_3, x_2, x_1, x_0) = \Sigma\,(0, 1, 3, 5, 7, 8, 10, 12, 15, 16, 17, 21, 24, 26, 28, 29, 30, 32, 33, 34, 35, 37, 39, 40, 42, 45, 46, 49, 50, 54, 55, 58, 59, 60, 61, 62, 63) \quad \ldots \quad (1)$$

We shall find all the cells incident with each vertex and also the weights of the vertices.

The above is a six-variable function. In Svoboda's method the function is first projected in a modified Veitch diagram. The different cells are found by using six contact grids and the weight of each vertex is found by using six directional grids (Svoboda, 1956 and Choudhury, 1959). Following McClusky (McClusky, 1956) we shall adopt a method in which the ideas of cells and weight can be directly incorporated and which can be easily extended to cover cases involving more than six variables.

To start with Table II is prepared as follows:

The decimal numbers corresponding to the vertices of the given function [Eq. (1)] are entered in column (a) in groups (indicated by separations) having increasing index values viz., $r = 0, 1, \ldots 6$.

The combinations entered in column (b) are selected from column (a) taking two numbers having index values r and $r+1$ respectively when

$$M_{r+1} - N_r = 2^k \quad \ldots \quad (2)$$

where $k = 0, 1, 2 \ldots$ and M_{r+1}, N_r are numbers belonging to the groups having index values $r+1$ and r respectively. The terms in column (b) will then show all possible 1-cells present in the given function. The difference expressed by Eq.(2) is entered within brackets. Thus 1 and 3 form a 1-cell, but 1 and 10 will not form a 1-cell.

The combinations in column (c) are derived from column (b), taking two terms from any two consecutive groups (i.e. on two sides of a separation line) when their first difference (Eq.(2)] tally and the second difference between leading numbers is again positive and equals 2^k, $(k = 0, 1, 2, \ldots)$, these differences being indicated in brackets. The terms in this column show all possible 2-cells present in the given function. We need only enter cells whose vertices form an increasing sequence of decimal numbers.

Similarly, column (d) has been prepared from column (c) when both the first and second differences tally and the third difference between leading numbers is again 2^k. The term in column (d) shows the only 3-cell present in the function.

Check marks are placed at any stage of combination when cells of a given k-value combine to form a cell of next higher order. We also check mark the cells which will give rise to alternative modes of formation of any higher order k-cell. Thus 1-cells 0, 1 (1) and 16, 17 (1) and also 0, 16 (16) and 1, 17 (16) are check marked since they form the 2-cell 0, 1, 16, 17 (1, 16). The unchecked cells are called the basic cells or the prime implicants of the given function.

Column (b) of Table II contains all information about the neighbours and weight of each vertex. The number of times a given vertex combine in this column is equal to its weight and the companions are its neighbours. Thus the vertex 1 has neighbour 0 in the top group and 3, 5, 17, 33 in the group just below the top, and we need not look down the column after the 1-cell 1, 33 (32), because 1 can never occur below this term. Hence weight of 1 is 5. Thus one can quickly compute the weight of each vertex, and find its neighbours, as listed in Table III.

THE MINIMISATION METHOD

The determination of minimum sum essentially consists of selecting a minimum number of basic cells so that their sum gives the specified output for all combinations of input variables.

TABLE II

Determination of cells

	(a) ✓	(b) ✓	(c)	(d)
$r = 0$	0 ✓	0,1(1) ✓	0,1,16,17(1,16)—*W*	1,3,5,7,33,35,37,39(2,4,32)—*A*
	1 ✓	0,8(8) ✓	0,1,32,33(1,32)—*V*	
	8 ✓	0,16(16) ✓	0,8,16,24(8,16)—*U*	
$r = 1$	16 ✓	0,32(32) ✓	0,8,32,40,(8,32)—*T*	
	32 ✓	1, 3(2) ✓	1,3,33,35,(2,32) ✓	
	3 ✓	1, 5(4) ✓	1,3,5, 7,(2,4) ✓	
	5 ✓	1,17(16) ✓	1,5,17,21(4,16)—*S*	
	10 ✓	1,33(32) ✓	1,5,33,37(4,32) ✓	
	12 ✓	8,10(2) ✓	1,17,33,49(16,32)—*R*	
$r = 2$	17 ✓	8,12(4) ✓	8,10,24,26(2,16)—*Q*	
	24 ✓	8,24(16) ✓	8,10,40,42(2,32)—*P*	
	33 ✓	8,40(32) ✓	8,12,24,28(4,16)—*O*	
	34 ✓	16,17(1) ✓	32,33,34,35(1,2)—*N*	
	40 ✓	16,24(8) ✓	32,34,40,42(2,8)—*M*	
	7 ✓	32,33(1) ✓	3,7,35,39(4,32)	
	21 ✓	32,34(2) ✓	5,7,37,39(2,32)	
	26 ✓	32,40(8) ✓	10,26,42,58(16,32)—*L*	

TABLE II—(*contd.*)

Determination of cells

	(*a*) √	(*b*) √	(*c*)	(*d*)
$r = 3$	28 √	3,7(4) √	24,26,28,30(2,4)—*K*	
	35 √	3,35(32) √	33,35,37,39(2,4) √	
	37 √	5,7(2) √	34,42,50,58(8,16)—*J*	
	42 √	5,21(16) √	26,30,58,62(4,32)—*I*	
	49 √	5,37(32) √	28,29,60,61(1,32)—*H*	
	50 √	10,26(16) √	28,30,60,62(2,32)—*G*	
	15 √	10,42(32) √	42,46,58,62(4,16)—*F*	
	29 √	12,28(16) √	50,54,58,62(4,8)—*E*	
	30 √	17,21(4) √	54,55,62,63(1,8)—*D*	
	39 √	17,49(32) √	58,59,62,63(1,4)—*C*	
$r = 4$	45 √	24,26(2) √	60,61,62,63(1,2)—*B*	
	46 √	24,28(4) √		
	54 √	33,35(2) √		
	58 √	33,37(4) √		
	60 √	33,49(16) √		
	55 √	34,35(1) √		
$r = 5$	59 √	34,42(8) √		
	61 √	34,50(16) √		
	62 √	40,42(2) √		
$r = 6$	63 √	7,15(8)—*b*		
		7,39(32) √		
		21,29(8)—*a*		
		26,30(4) √		
		26,58(32) √		
		28,29(1) √		
		28,30(2) √		
		28,60(32) √		
		35,39(4) √		
		37,39(2) √		
		37,45(8)—*Z*		
		42,46(4) √		
		42,58(16) √		
		50,54(4) √		
		50,58(8) √		
		29,61(32) √		
		30,62(32) √		
		39,55(16)—*Y*		
		45,61(16)—*X*		
		46,62(16) √		
		54,55(1) √		
		54,62(8) √		
		58,59(1) √		
		58,62(4) √		
		60,61(1) √		
		60,62(2) √		
		55,63(8) √		
		59,63(4) √		
		61,63(2) √		
		62,63(1) √		

TABLE III

Determination of weight and neighbours

Vertex	Neighbours	Weight
0	1, 8, 16, 32	4
1	0, 3, 5, 17, 33	5
8	0, 10, 12, 24, 40	5
16	0, 24, 17	3
32	0, 33, 34, 40	4
3	1, 7, 35	3
5	1, 7, 21, 37	4
10	8, 26, 42	3
12	8, 28	2
17	1, 16, 21, 49	4
24	8, 16, 26, 28	4
33	1, 32, 35, 37, 49	5
34	32, 35, 42, 50	4
40	8, 32, 42	3
7	3, 5, 15, 39	4
21	5, 17, 29	3
26	10, 24, 30, 58	4
28	12, 24, 29, 30, 60	5
35	3, 33, 34, 39	4
37	5, 33, 39, 45	4
42	10, 34, 40, 46, 58	5
49	17, 33	2
50	34, 54, 58	3
15	7	1
29	21, 28, 61	3
30	26, 28, 62	3
39	7, 35, 37, 55	4
45	37, 61	2
46	42, 62	2
54	50, 55, 62	3
58	26, 42, 50, 59, 62	5
60	28, 61, 62	3
55	39, 54, 63	3
59	58, 63	2
61	29, 45, 60, 63	4
62	30, 46, 54, 58, 60, 63	6
63	55, 59, 61, 62	4

McClusky's method consists essentially of drawing a prime implicant table, selecting the basis rows, then ruling out each row which is covered by another. The first step is now repeated and the procedure continued until all the states are included or a cyclic prime implicant chart results. Then a trial repetition process is followed to obtain the minimal sum.

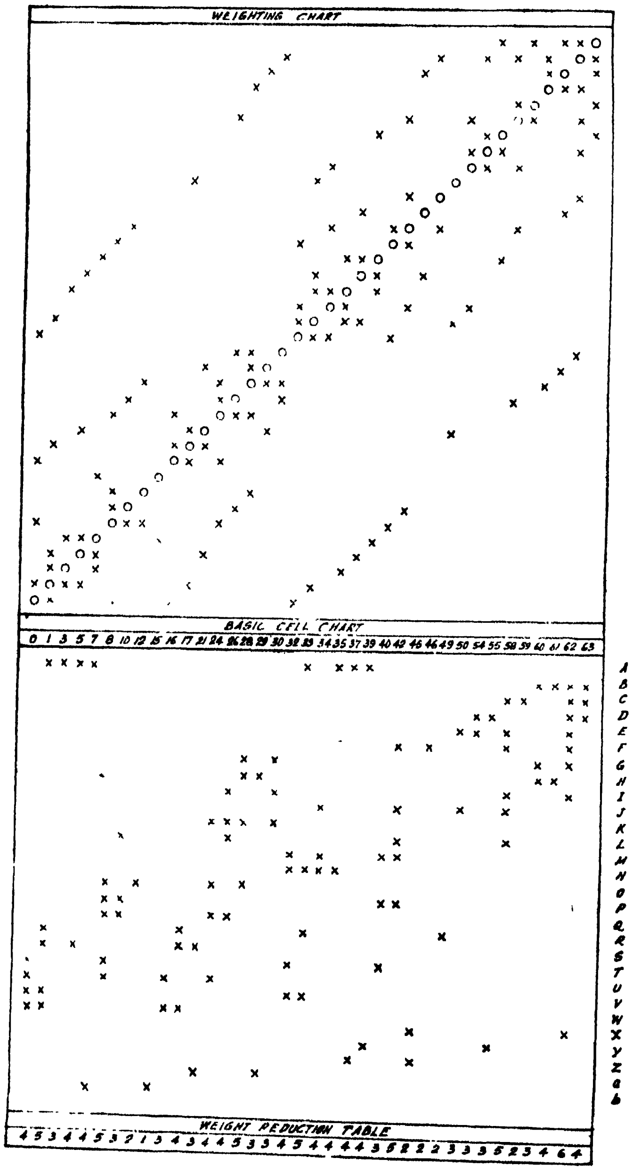

Fig. 1. The basic-cell chart, the weighting chart and the weight reduction table for the function expressed by Eqn. (1).

Svoboda follows a few methodical steps realised by applying the contact grids and directional grids on a modified Veitch diagram. We shall adapt these steps to McClusky's chart, with some obvious advantages. The method may be illustrated by taking the example of minimising the function given in Eq. (1).

A basic cell chart, which is what McClusky calls "the prime implicant table", is first drawn.(Fig. 1). The columns carry at their heads the decimal numbers corresponding to all the vertices which are contained in the expression of the function and the horizontal rows correspond to basic cells. The vertices which combine to form a basic cell are cross-marked at the intersections with the horizontal row representing the particular basic cell. Thus the basic cell Z has been formed by combining 1, 3, 5, 7, 33, 35, 37, 39 and crosses in the A row are placed under these columns only and so on for all other rows.

We shall now introduce one new chart and a table. Over the basic cell chart, a weighting chart is placed which depicts all the neighbours of each vertex and hence determines the weight of the latter. This chart is drawn from the data in Table III or directly from column (b) of Table II or, if anybody prefers, with the help of a Karnaugh map. Small circles are entered over each of the vertex of the given function in successive horizontal lines so that the circles are located over a diagonal line. Crosses entered in the horizontal line are the neighbours of the vertex represented by the circle in that horizontal line. Thus, crosses corresponding to 1, 8, 16, 32, are the neighbours of the circle representing the vertex 0. The weighting chart has the interesting property that it is symmetrical about the diagonal line.

Below the basic cell chart, a weight reduction table is formed. In the first row of this table, weights of all vertices computed from the weighting chart are entered under corresponding columns.

We shall now proceed to utilise Svoboda's methodical steps (Svoboda, private communication) to obtain the minimal form.

FIRST STEP

The terms which are essential for inclusion in all possible minimal forms satisfy the theorem:

"Theorem I: The sufficient condition for inclusion of a term T in (any of) the minimal forms of the function is the incidence of the corresponding k-cell t with a vertex V of weight k".

We begin with cells having smallest weight. If the weight of a vertex is zero or one, one always obtains a term satisfying the above condition. In Fig. 3 there is no vertex with weight $k = 0$. The vertex 15 has weight $k = 1$ and there is the 1-cell b(7, 15) in row (b), which therefore is an essential term. Again, the vertices 12, 46, 49, 59 have weights $k = 2$, and to each of them is incident one

2-cell in the rows O(8, 12, 24, 28), F(42, 46, 58, 62), R(1, 17, 33, 49) and C(58, 59, 62, 63) respectively, and hence they represent essential cells. But with the vertex 45 having weight $k = 2$, there is no 2-cell incident and hence there is no essential term corresponding to this vertex. The vertex 3 has weight $k = 3$ and there is a 3-cell incident with this in the row A(1, 3, 5, 7, 33, 35, 37, 39), which is therefore essential. No other vertex of weight $k = 3$ satisfies the sufficient condition of

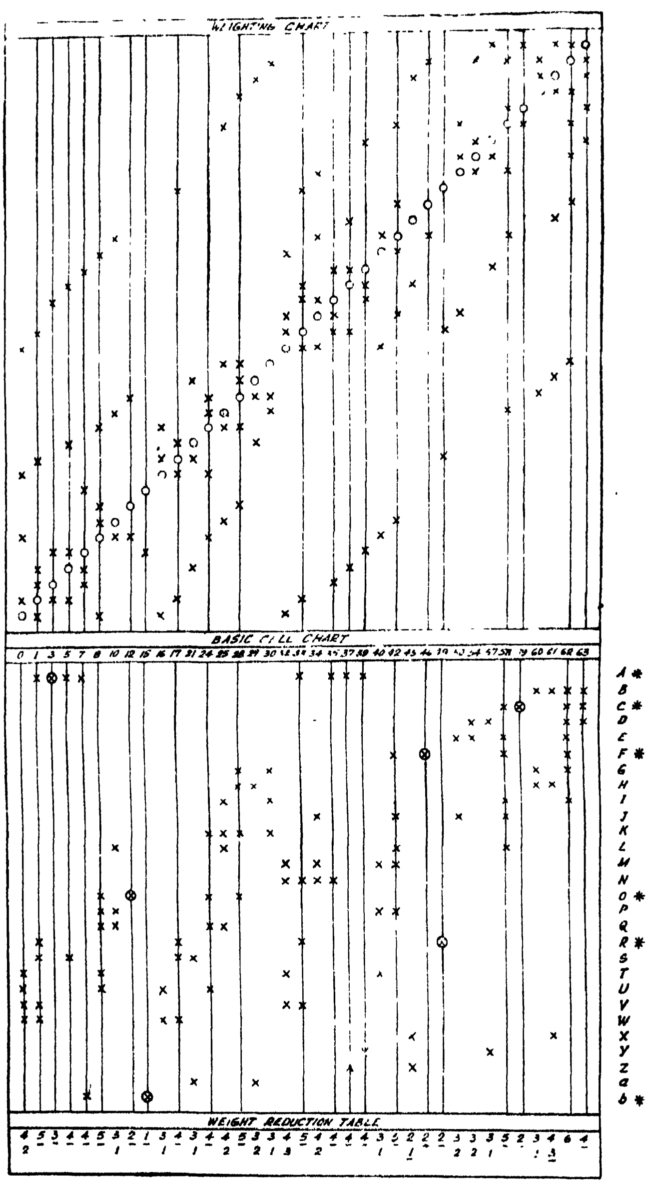

Fig. 2. The form of the charts after the first methodic step.

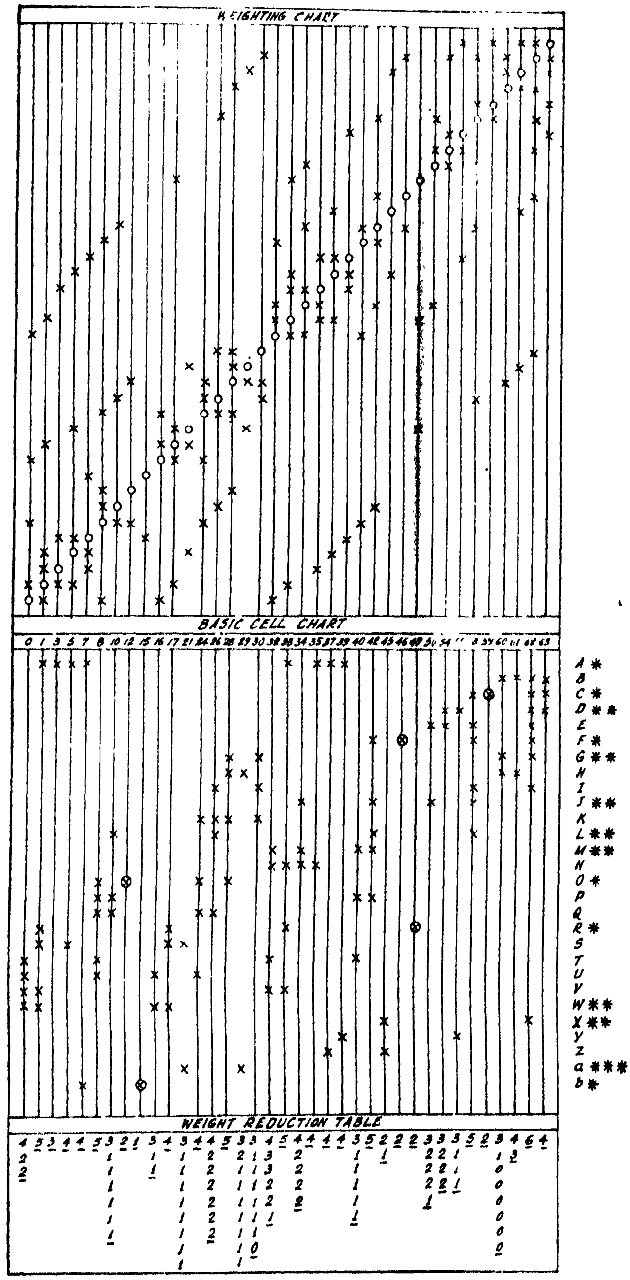

Fig. 3. The form of the charts after the second methodic step.

Theorem I. We do not require to look into the vertices of weights $k = 4, 5, 6$, since it is evident from Table II, that for the function in question, there is no cell of order 4, 5, or 6.

Thus, by the first methodic step we have selected rows A, C, F, O, R, b, marked by asterisk at the extreme right of the rows (Fig. 2). It is interesting to note that this step is identical with the first step in McClusky's method e.g., finding the columns which contain only one cross and selecting the rows as the basis rows in which these crosses occur.

All the columns in which the essential rows have entries are lined out, because the vertices corresponding to these columns have become bound to the chosen configuration of cells in the first step. The lining out is also extended to the weighting chart so that the free neighbours of the free vertices become apparent (see Fig. 2).

Definition : The free weight W of a vertex V_0 is equal to the number of 1-cells belonging to the body of the Boolean function incident with the vertex V_0 and incident with W free vertices $V_1, V_2 \ldots V_W$. The vertices $V_1, V_2 \ldots V_W$ are called free neighbours of the vertex V_0.

The reduced or the free weights of free vertices are now easily computed by counting the number of unruled crosses occurring in the horizontal line for any vertex. Thus for the free vertex 0, we find the number of unruled crosses is equal to 2 corresponding to the free neighbours 16 and 32. Hence the reduced weight of the vertex is 2. The weighting chart, therefore, not only enables to compute the reduced weight, but also shows which neighbours of a particular vertex are still free.

The reduced weight of each free vertex is computed and entered in the second row of the weighting table, under corresponding columns of the vertices. Each of the vertices which is bound in the first step is marked in this table by putting a bar under the number representing its weight. The appearance of the charts is shown in Fig. 2.

SECOND STEP

The second step is to select those terms which might be included in at least one of the minimal forms. For this we utilise the theorem :

Theorem II : A sufficient condition for a term T_{n+1} to be included in at least one of the minimal forms is satisfied when all the following propositions are true :

(1) The partial form $F_n = T_1 + T_2 + \ldots + T_n$ (corresponding to the body of all bound vertices built from cells $t_1, t_2 \ldots t_n$) has been included in one of the minimal forms.

(2) There is *a* K-cell t_{n+1} incident with a free vertex V_0, with all free neighbours $V_1, V_2 \ldots V_W$, with free vertices $V_{W+1}, V_{W+2}, \ldots V_{W+p}$ and with any number of bound vertices ($W \leqslant K$ being the free weight of the vertex V_0).

(3) There is no k-cell incident with the V_0 and with another free vertex not belonging to the set $V_0, V_1 \ldots V_{W+p}$.

(4) All k-cells incident with V_0 have $k \leqslant K$.

(5) The term T_{n+1} corresponds to the K-cell t_{n+1}

In Fig. 4, the free vertex 45 has free weight $W = 1$. It will be seen that there is a K-cell $X(45, 61)$ satisfying the sufficient conditions of Theorem II. Columns 45 and 61 are lined out and the lining out is extended to the weighting chart.

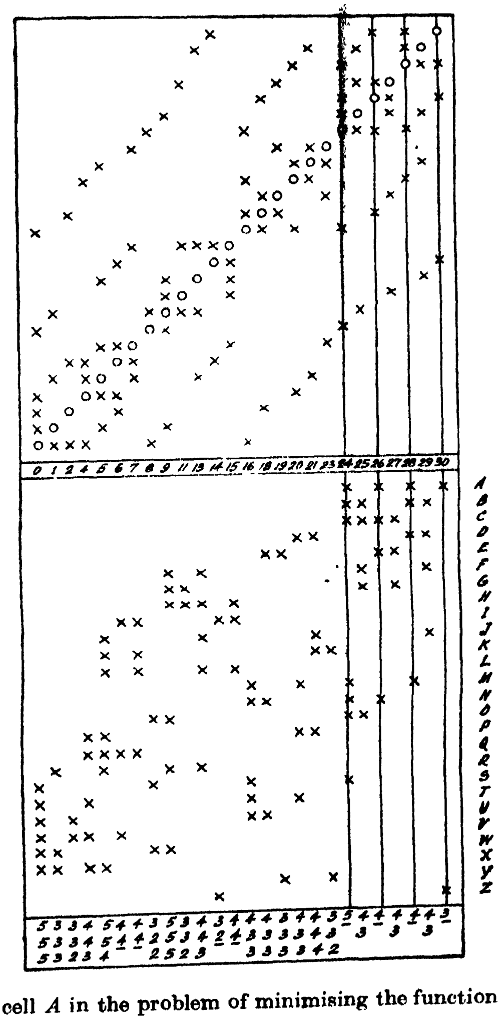

Fig. 4. Selection of cell A in the problem of minimising the function expressed in Eq. (iv).

Now, due to ruling out 61, weights of 60 and 29 are reduced further by unity. The weights for other free vertices remain unchanged. The free-weights are now written in the third row of the weighting table (Fig. 3).

The vertex 60 has free weight $W = 0$, but no cell incident on it (e.g., B, G, H)satisfies the sufficient condition of Theorem II, because with B, the vertex 30 of G or vertex 29 of H are not covered. Similar arguments apply to cells G or H. From the vertices having free weight $W = 1$, are incident 2-cells U and W with 16; both of these take all the free neighbours—here 0 only—of 16. Also $K = 2 > W = 1$; all k-cells have $k = K$, so that we can take either. But for vertex 10, cell L takes its only free neighbour 26, but there is a cell P incident with 10 containing the vertex 40 which is not included in cell L; hence L cannot be included. The vertex 21 has $W = 1$. The cell 'a' incident with 21 takes its only neighbour 29, its $K = 1 = W$, but there is a k-cell S having $k = 2 > K$; thus, it cannot be included. We can check similarly why cells incident with 29, 30, 40 for which $W = 1$, do not satisfy the conditions of Theorem II. But cell D incident with 55 having free weight $W = 1$, can be included.

Let us take W. The reduced weights of free vertices are shown in the fourth row of the weighting table.

Now, vertex 60 of $W = 0$, cannot still be included. With vertices 10, 21, 29, 30, 40, of weight $W = 1$, there is no cell incident satisfying conditions of Theorem II. With vertex 55 of weight $W = 1$ is incident cell D which takes 55's only neighbour 54 and for this all $k < K$; so row D is selected. The vertices 54, 55 are bound thereby. The weights of free vertices are computed and written in the fifth row of the weighting table. Another cell Y incident with 55 cannot be included. In the same way row J incident with 50 is selected and the free weights of the free vertices are reduced and entered in the sixth row of the weighting table. Next, row M incident with 32 is selected. The resulting reduced weights of the free vertices are shown in the seventh row of the weighting table. Now, two cells L and Q incident with 10, satisfy sufficient conditions for inclusion. We take L and reduce the weights of the free vertices. Note that when vertex 40 was unbound, L could not be included as an acceptable cell incident with 10). Next, incident with vertex 30 of $W = 0$, three cells, names, G, I, K satisfy sufficient conditions of inclusion. Let us take G, since it also bounds the free vertex 60 of free weight $W = 0$. The cells selected at the second step are indicated by two asterisks.

The form of the chart at this stage is shown in Fig. 3. Two vertices, 21 and 29 are still free, and Theorem II is not applicable to them.

THIRD STEP

When both theorems I and II cannot be applied any further, the general procedure will be to start with a vertex having smallest free weight and pick out the minimal form by trying all possible cells covering the neighbours of the vertex, and completing the minimal form for each trial, by a repetition of the second methodic step. In case, the second step made after a trial does not include all the free vertices, a second trial is to be made, repeating a third step followed by another repetition of a second step. From all the trials, we call the one minimal which introduces minimum number of operations.

In the example that we have chosen, this step is trivial. The vertices 21 and 29 are selected by cell 'a' marked by three asteriks and there is no other alternative.

Thus one of the minimal forms of the Boolean Function given by Eq. (i) is

$$\begin{aligned} f = A+C+F+O+R+b & \quad \text{(first step)} \\ +X+W+D+J+M+L+Q & \quad \text{(second step)} \\ +a & \quad \text{(third step)} \qquad \ldots \quad \text{(iii)} \end{aligned}$$

To write the algebraic equivalent of a basic cell, for example, of the cell F given by 42, 46, 58, 62(4, 16) we write the leading number 42 in binary form, eliminate variable x_k whenever a difference 2^k appears in the bracket, and for the remaining digits '1' stands for an unprimed variable and '0' for a primed variable.

2^k	k	x_5	x_4	x_3	x_2	x_1	x_0		F
4, 16	2, 4	1	ϕ	1	ϕ	1	0	$= 42$	$x_5x_3x_1x'_0$

and so on for the other cells.

We can also obtain alternative minimal forms by interchanging cells without increasing the number of operations. Thus, we have seen that interchange is possible between W and U or L and Q, which will produce three more minimal forms. We also note that if L, M and D are not eliminated, we could interchange J with E producing a fourth minimal form and so on.

MODIFIED THIRD STEP AND CYCLIC BASIC CELL CHART

The third step of trial-repetition in both Svoboda's and McClusky's method is not very smooth. In fact when the cyclic basic cell chart (*) is considerably complex, there is no way out to break through the structure in McClusky's method. In Svoboda's method, however, there is one clue: start with vertices of smallest weights. At this point a modification might be introduced which will reduce the number of trial repetitions to a great extent. The modification is as follows:

(a) Start always with a vertex having smallest weight.

(b) Select the cell incident with the vertex which includes all the neighbours of the vertex. In case there are more than one such cells, selection of any one of them will suffice. But, if cells are incident with the chosen vertex which include not only the neighbours but other non-neighbour free vertices, select that row which includes the maximum number of non-neighbour free vertices. If this

*A basic cell chart is here said to be cyclic when with none of the vertices is incident a cell satisfying the conditions of both Theorem I and II.

number be the same for more than one cell, selection of any one of them will suffice.

(c) If a cell incident with a chosen vertex does not include all neighbours, select that cell which covers the maximum number of neighbours, and if there are more than one such cell, selection of any one of them will suffice. But, if there are cells incident with the vertex which not only includes the same maximum number of neighbours, but also some other non-neighbour free vertices, select that cell which includes the maximum number of non-neighbour free vertices. If this latter number be the same for more than one cell, selection of any one of the mwill suffice.

After each selection of a cell in (b) or (c) , reduce the weights of the free vertices and repeat the procedure until all the vertices are bound.

To illustrate the above procedure we choose a problem which can not be readily solved by McClusky's trial repetition method. McClusky gave an approximate solution of this problem by his method of selecting consistent-row set (McClusky, 1956).

The problem is to obtain the minimum sum of

$$\begin{aligned} f(x_3, x_2, x_1, x_0) = (&0,\ 1,\ 2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 11,\\ &13,\ 14,\ 15,\ 16,\ 18,\ 19,\ 20,\ 21,\\ &23,\ 24,\ 25,\ 26,\ 27,\ 28,\ 29,\ 30) \end{aligned} \qquad \ldots \quad \text{(iv)}$$

The basic-cell chart, the weighting chart, and the weight reduction table are shown in Fig. 4. This is a cyclic basic cell chart. Incident with the vertex 30 of weight $W = 3$, there are two cells A and Z; neither of them covers all the neighbours of 30 viz., 14, 26, 28 so that criterion (b) is not applicable. Since A takes two neighbours and also a non-neighbour vertex 24, while Z only one. We select A [criterion (c)]. Columns heading 24, 26, 28. 30 are lined out, lining out extended to the weighting charts, reduced weights of free vertices are written in the second row of the weighting table (See Fig. 4). Now, the vertices 8 and 14 have free weight $W = 2$ and to both of them criterion (b) is applicable. The cell I incident with 14 takes two neighbours of 14 viz., 6, 15 and one non-neighbour free vertex 7. Also, cell W incident with vertex 8 takes the two neighbours of 8-0, 9 and one non-neighbour free vertex 1. Hence any one of the cells will do. Let us take I. The bound columns are lined out. Weights of the free vertices are reduced and entered in the third row of the weighting table. Now, vertices 2, 8, 11 have free weights $W = 2$. We can take either U or W or G at this stage, since all of them cover the free neighbours of 2, 8, 11 respectively and each has one non-neighbour free vertex, viz., 16, 1, 27 respectively. It is to be noted that vertex 23 also has free weight $W = 2$, but there is no cell incident covering all the free neighbours. Hence it is not considered. Let us take G and reduce the free weights.

The form of the charts after selection of A, I and G is shown in Fig. 5. Now, the vertex 8 has minimum free weight $W = 1$. We can select either W or S

[criterion (b)]. Let us take W and reduce weights of free vertices. Selection of U is now unique [criterion (b)]. Weights of free vertices are reduced and shown

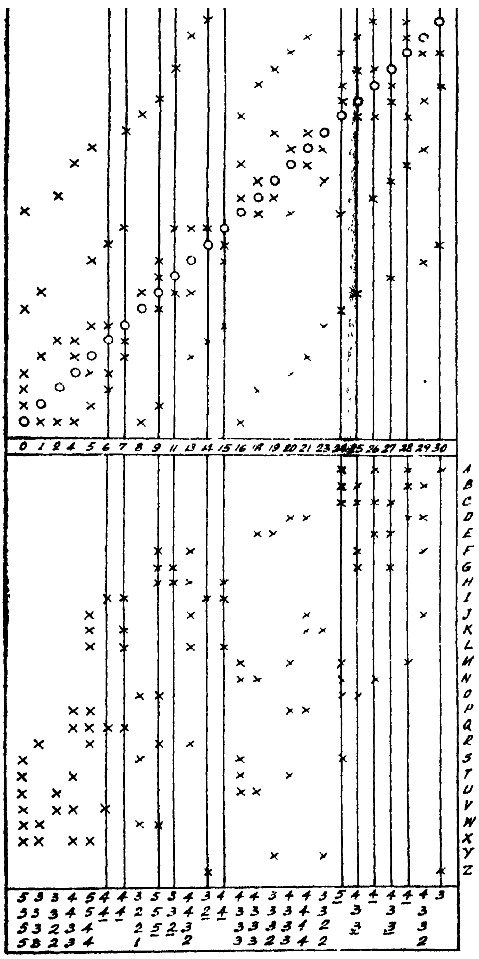

Fig. 5. The form of the charts after selection of cells A, I, G.

in the sixth row of the weighting table. Selection of Y is then unique. Free weights are reduced and shown in the seventh row. Then vertices 4, 13, 20, 29 have weights $W = 2$. Both 13 and 29 are covered by J and this cell besides providing for the neighbours, also include one non-neighbour free vertex. The cell P incident with 20, also satisfies similar conditions; let us, however, take J. This reduces free weights of 4 and 20 to $W = 1$. The condition of the charts is now shown in Fig. 6. We can include the vertices 4 and 20 at a time either by T or P. Let us take T. The minimal form, therefore, is

$$f = A+I+G+W+U+Y+J+T \qquad \dots \quad (v)$$

There are situations when all the alternative minimal forms can be obtained directly from one minimal form by interchange of variables or by priming variables or by both. For example, let us take the four variable function

$$f(x_3, x_2, x_1, x_0) = (0, 1, 3, 5, 8, 10, 11, 13, 15) \qquad \ldots \quad \text{(vii)}$$

The basic-cell chart, weighting chart and weight reduction table (for two trials) are shown in Fig. 7. If we start with vertex 0 and select cell A applying criterion (c) we shall obtain the minimum sum as

$$f_1 = A+F+G+E+ \genfrac{}{}{0pt}{}{J}{I} \qquad \text{(viii)}$$

The sequence of operation is evident from weighting Table I. If, instead of A, we start with cell B, we shall obtain the solution (weighting Table II)

$$f_2 = B+H+C+G+ \genfrac{}{}{0pt}{}{J}{I} \qquad \text{(ix)}$$

Now, f_2 can be directly obtained from f_1 by priming x_1 and x_3 and interchanging, since the above operation will leave the function unchanged; vertices 0, 1 are changed with vertices 10, 11 respectively so that cells A, E, F are changed to H, B, C respectively. This property of the function is called group invariance.

CONCLUSION

A method of minimisation of Boolean Functions in the form of a minimum sum has been presented in this paper by applying the methodical stèps of Svoboda on McClusky's chart with the help of a weighting chart and a weight reduction table. The merit of this method is that it can be conveniently extended to cover cases involving more than six variables whereas the application of Svoboda's method to such cases becomes considerably inconvenient. Furthermore, in this method the basic cells incident with any vertex are easily obtained by looking along the column representing the vertex, whereas in Svoboda's method they will have to be searched out by several trials involving tossing and turning of the grids having different combinations.

A modification has been suggested on the third step of trial-repetition, which is particularly useful for minimising functions which form into cyclic basic cell charts.

The introduction of weighting chart has greatly systemetised the advantages of the methods of both Svoboda and McClusky.

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